# Sphere Packing Made Simple

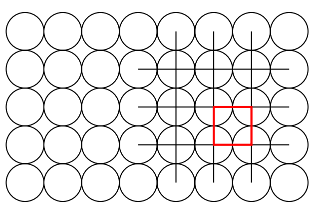
Imran Parvez Khan

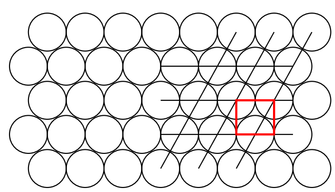
**Abstract**

The year 2022 was special in Mathematics because the Fields Medal prize (that is awarded to Mathematicians under 40, every four years) was awarded. Among other recipients was the Ukrainian Mathematician Maryna Viazovska, who became the 2nd woman to win this prestigious award (the first being Maryam Mirzakhani receiving Fields Medal in 2014). Maryna Viazovska has solved the problem of Sphere Packing in higher dimensions. This note is a brief exposition of her work.[1]

## Introduction

In my childhood days, when I used to get bored in the classroom, I used to place coins on the desk. The two most common arrangements that I came up with are shown below. As a Mathematician I am posed with the question that





which of these arrangements is better, that is, which one packs the circles more tightly than the other. To answer this, I would like to first give the definition of a lattice as follows.

**Definition 1.** *A periodic arrangement of points is called a lattice.*

In order to determine the underlying lattice for each of the arrangements shown in the figure above, let us draw the vertical and horizontal lines passing through the centres of the circles in the arrangement on the left (shown partly). What we obtain as a result is known as the *rectangular lattice*. In the arrangement on the right, the vertical lines get replaced by lines with a 60 degrees slope (shown partly). For this reason, the underlying lattice in this case is called the *hexagonal lattice*.

[1]Based on a talk presented at The Black Hole, Islamabad, Pakistan on 17-01-2023.

Since the arrangements are infinite, we will only consider a finite region of these arrangements called the fundamental region, and see how many circles are contained in that fundamental region. So that the density of lattice packing is defined as follows.

**Definition 2.** *Density of lattice packing* $=\frac{\textit{Area occupied by the circles}}{\textit{Area of the fundamental region}}$

In both the packings, we are assuming the radius=1/2. The fundamental region for the rectangular lattice is a square with unit area. And the area of the circle is $\pi/4$. Thus the density of rectangular lattice packing is $\pi/4 = 0.78$. We may interpret that in a rectangular arrangement of circles 78% area is being occupied by the circles. In the case of hexagonal packing, the fundamental region is a rectangle, and one will have to use Pythagoras' theorem in order to determine one of the sides of the rectangle. The density then turns out to be $\pi/2\sqrt{3} = 0.90$, i.e. 90% of the area is being occupied by the circles. This brief demonstrations helps to show that in two dimensions the "best" way to pack circles is by using the hexagonal lattice. One can try drawing larger fundamental regions and they will correspondingly contain more circles, but the density will remain the same. Some readers may object that the fundamental region in case of hexagonal packing should be a parallelogram. Again it is a matter of checking that a parallelogram and a rectangle of same area will contain same amount of circles, thus keeping the density unchanged.

The corresponding problem in three dimensions is called the sphere packing problem. The two-dimensional case prepares us to understand the sphere packing problem in any dimension. Notice that two or more cubes can fill a volume of space without leaving any gaps between them, but spheres cannot. So the sphere packing problem asks which lattice packing arranges the spheres in a given volume without overlapping such that the density is optimal. We would also like to give the following definition of lattice packing.

**Definition 3.** *A lattice packing of spheres centers the spheres at the lattice points.*

# Sphere Packing Problem in Higher Dimensions

In three dimensions, the sphere looks like a ball, but what does the corresponding object look like in four or five or higher dimensions? It is better to settle this matter before we proceed further. A similar problem is faced when discussing higher dimensions. We usually label the three dimensions with alphabets $x, y, z$. But the alphabets are only 26, whereas it is common in Mathematics to consider thousand or many more dimensions. So a better practice is to use subscripts and label dimensions such as $x_1, x_2, x_3, \ldots$. Similarly here, due to lack of jargon, we use the term sphere in its full generality. In three dimensions a sphere looks like a ball. But since we cannot imagine higher dimensions, so we detach the image of a ball with the "sphere", and call the corresponding object in any dimensions also a sphere. Similarly we understand what is meant by volume in three dimensions, so we generalise the same notion to dimensions four and above.

Now replacing circle by sphere and area by volume in Definition. 2, we can use that for finding the density of lattice packing in any dimension. Denoting the volume of an $n-$dimensional sphere by $V_n$, we can find it using the following expressions, $V_n = \frac{\pi^{n/2} r^n}{(n/2)!}$ (for $n$ even) and $V_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} r^n$ (for $n$ odd). It is fun and simple to use these formulas as all you need to input is the desired $n$. For $n = 1, 2, 3$ we obtain the familiar looking expressions, $V_1 = 2r, V_2 = \pi r^2, V_3 = \frac{4}{3}\pi r^2$. Let me use this chance to "extend" the generalisation of notions of sphere and volume to lower dimensions also. For instance note that $V_2 = \pi r^2$, which is the area of the circle, because two-dimensional volume is area. Similarly the one dimensional sphere is a straight line, and its volume is simply its length. For the reader who might be wondering what then is the sphere packing density in one dimension, imagine that you have got one dimension, and several match sticks to arrange. The match sticks will fit in perfectly in the one-dimensional space hence the sphere packing density in one dimension is 100%. Maryna Viazovska (Fields Medal recipient 2022) has solved the sphere packing problem for dimensions 8 (independently) and 24 (with collaborators). I only focus on the eight-dimensional case in this note. Putting $n = 8$ in the formula above,

$$V_8 = \frac{\pi^4 r^8}{4!}.$$

Recall that in the two-dimensional case, we analysed the underlying lattice of the packing, in order to use Definition. 2. Similarly in eight dimensions, we must know the underlying lattice. The lattice in this case is called the $E_8$ lattice. It is a very deep topic in both Number Theory and Lie Theory, so I will suffice with a working definition,

**Definition 4.** *$E_8$ lattice is an even sum lattice.*

In order to discuss what is meant by an *even sum lattice* above, I would first like to come to a convenient dimension such as three. Now here if the ordered pairs in the lattice are such that the sum of the components is an even number, then the lattice is called an even sum lattice. Following are some example points (1,1,2), (2,4,6), (3,1,0), (1,1,0),. . . . (Of course in eight dimensions, each point will have eight components such that their sum is even.) The length of the vectors emanating from (0,0,0) to the above mentioned points is $\sqrt{6}, \sqrt{56}, \sqrt{10}, \sqrt{2}$ respectively. Hence we notice that in general the length of vectors in an even sum lattice will always be of the form $\sqrt{2k}$, where $k$ is a positive integer. The same is true for $E_8$ (see References). Which tells us that the shortest vector present in $E_8$ is of length $\sqrt{2}$.

The significance of this number is that since a lattice packing of spheres centres the spheres at the lattice points, we cannot allow the sphere to have its centre at, say, the midpoint of two lattice points. So only the lattice points will serve as the centres for the spheres, and since the spheres cannot overlap, thus only two spheres can occupy space between any two lattice points. Thus the radius of the sphere will be half of the shortest vector of the lattice. Thus for $E_8$, the radius of the sphere is $(1/2)(\sqrt{2}) = 1/\sqrt{2}$. Substituting $r = 1/\sqrt{2}$ in the expression for $V_8$, we get

$$V_8 = \pi^4/384.$$

We also need the volume of the fundamental region to find the density of the lattice packing. For dimension 8, the volume of the fundamental region turns out to be unity (In fact, $E_8$ is called a unimodular lattice). This can be understood by realising that the volumes can be calculated using determinants, i.e., if one knows the basis vectors of the lattice, then calculating the volume is not difficult. (For example, use internet to find the area of parallelogram formed by vectors (1,3) and (4,2), and then calculate using determinant, the answer in both ways will be the same.) It is amusing that we neither know what an eight-dimensional ball looks like, nor what the eight-dimensional space looks like, but we know that the sphere packing density in eight dimensions is 25%.

## Work of Maryna Viazovska

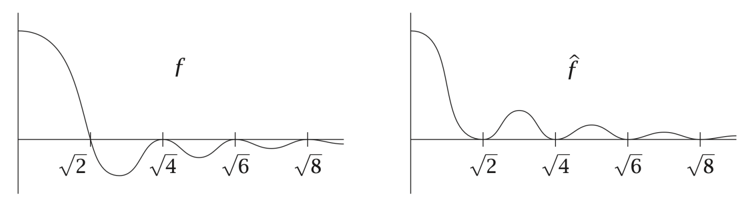
Now we can present the statement of the main theorem of Maryna Viazovska: the $E_8$ lattice achieves the optimal sphere packing density in eight-dimensional space, namely $\pi^4/384$.

Just as in the case of two dimensions, other choices for lattices are also possible in eight-dimensions. The corresponding densities can also be easily found. So what is the problem then? The problem is to prove that a certain lattice has the *optimal* density.

In order to prove that $E_8$ is the densest packing in eight dimensions, as claimed in the theorem of Maryna, the pioneering researcher in this field, Henry Cohn had proved the existence of a function, which if satisfied the following properties, then the statement of Maryna's theorem is true:
(i) $f(x) \leq 0$ for all $x \geq \sqrt{2}$,
(ii)$\hat{f}(y) \geq 0$ for all $y$.
Where $\hat{f}$ is the Fourier transform of the function $f$. Such a function was supposed to be impossible to find. And there is a reason for that pessimism. Let me explain this with the help of an example. Light is composed of different frequencies. In order to find how much amount of which frequencies is present in a particular light signal, we need the Fourier transform of the light signal. That is if the light signal is presented by the function $f(t)$, then its Fourier transform will be presented by $\hat{f}(\omega)$. Here $t$ is the time and $\omega$ is the frequency. In context of Fourier transform, a time interval and a frequency interval cannot simultaneously be precisely defined, mathematically $\Delta t\, \Delta\omega \geq \frac{1}{4}$. For instance, if $\Delta t = 1/2$, then $\Delta\omega$ cannot be less than 1/2. If $\Delta\omega = 1$, then $\Delta t$ cannot be less than 1/4, etc. This is called Uncertainty Principle, and is the obstacle in finding the function proposed by Cohn.

When Maryna was working on the construction of the required function, she simplified a calculation by discarding some terms. But to justify this step, she demanded the function to vanish at the lattice points because there is no harm in discarding terms that are 0 anyway. In summary, Maryna was confronted with several conditions; the function should have such and such values; its Fourier transform should have such and such values; and it must be zero at the lattice points. But what was the function that had all these features?

I will not present the detailed derivation of the function that satisfies these conditions, because that would involve Number Theory and Modular Forms. But I will show the graph of that function, and describe its basic characteristics visible directly from the graph. The graph of $f$ and $\hat{f}$ is presented below. The graphs touch the horizontal axis at certain points. These points coincide with the lattice points. So in the graph of $f$, not only the roots are brought about




by choice, but notice that at $x = \sqrt{2}$, the curve crosses through the horizontal axis, while at other roots, the curve only touches the horizontal axis and returns back. This is because the first condition in Cohn's theorem requires that $f \leq 0$ beyond $\sqrt{2}$. The root at $\sqrt{2}$ is called a single root. While the other roots are called double roots, and are different from single root in that they are not only the roots of $f$, but also of the derivative of $f$. Similarly $\hat{f}$ stays above the horizontal axis as per the second condition in Cohn's theorem.

Though the function involves modular forms, but there is a trigonometric factor multiplied to the original function that is significant, and is easy to understand. This factor is $-4\sin^2\left(\pi r^2/2\right)$. (It can also be visualised by feeding it in wolframalpha.com.) If one plots this as a function of $r$, the graph that emerges shares a few features with the graph of $f$. For instance the graph is zero at the lattice points of $E_8$. The graph lies below the horizontal axis for all values, due to the presence of minus sign.

## Acknowledgments

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